Probability Theory
2015/16 Semester IIb
Instructor: Daniel Valesin
Reexamination
5/7/2016
Duration: 3 hours

Name: ______________________
Student number: __________

This exam contains 9 pages (including this cover page) and 7 problems. Enter all requested information on the top of this page.

Your answers should be written in this booklet. Avoid handing in extra paper.

You are allowed to have two hand-written sheets of paper and a calculator.

You are required to show your work on each problem.

Do not write on the table below.

Problem	Points	Score
1	14	
2	10	
3	14	
4	14	
5	14	
6	14	
7	10	
Total:	90	

Standard Normal cumulative distribution function
The value given in the table is $F_X(x)$ for $X \sim \mathcal{N}(0,1)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

1. (a) (7 points) A deck consists of 52 distinct cards, among which 4 are aces. The deck is distributed randomly among 4 players, so that each player receives 13 cards. Find the probability that each player receives one ace.

   (b) (7 points) 40% of the boys in a certain village are polite and the remaining 60% are impolite. Polite boys open doors to elders 2/3 of the time, whereas impolite boys only do so half the time. Little Paul is seen opening the door to Mrs Marple, but not opening it to Mr Poirot. What is the probability that he is polite?

2. (10 points) Let $X \sim \text{Geometric}(1/2)$, $Y \sim \text{Geometric}(1/3)$ and $Z \sim \text{Geometric}(1/4)$ be independent. Find $\mathbb{P}(X = Y = Z)$ and $\mathbb{P}(X < Y < Z)$.

3. Suppose that we put together in an urn one ball with the number 1 written on it, two balls with the number 2 written on them, ..., n balls with the number n written on them. (The urn ends up containing $1 + 2 + \cdots + n$ balls). We then select a ball at random from the urn (that is, all the balls have the same probability of being chosen). Let Y_n be the number written on the chosen ball.

   (a) (7 points) Find $\mathbb{E}(Y_n)$. You may use the formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (b) (7 points) Show that, as $n \to \infty$, $\frac{Y_n}{n}$ converges in distribution to a continuous random variable Y with probability density function

$$f_Y(y) = \begin{cases} 2y, & \text{if } 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

4. Let $X_1, \dots, X_n$ be independent and identically distributed continuous random variables with finite expectation μ and variance σ^2. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) (7 points) Show that

   $$f_{\overline{X}_n}(x) = n f_{X_1+\cdots+X_n}(nx).$$

   (b) (7 points) Find the variance of $5\overline{X}_n - 4X_1$.

5. Let X and Y be independent random variables, both following the exponential distribution with parameter 1.
   (a) (7 points) Find the joint probability density function of $U = X - Y$ and $V = X + 2Y$.
   (b) (7 points) Find $f_{U|V}$.

6. (a) (7 points) Let

$$\begin{aligned}
&A_1^{(1)},\ A_2^{(1)},\ A_3^{(1)}, \dots,\\
&A_1^{(2)},\ A_2^{(2)},\ A_3^{(2)}, \dots,\\
&\qquad\vdots\\
&A_1^{(k)},\ A_2^{(k)},\ A_3^{(k)}, \dots
\end{aligned}$$

be k sequences of events in a probability space. Show that, if

$$\mathbb{P}(A_n^{(i)}) \xrightarrow{n\to\infty} 1 \quad \text{for } i = 1, 2, \dots, k,$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^{k} A_n^{(i)}\right) \xrightarrow{n\to\infty} 1.$$

(b) (7 points) Let $X_1,\ X_2,\ X_3,\ \dots$ be a sequence of independent and identically distributed discrete random variables, all with the same probability mass function f. Show that, for any real numbers $x_1, \dots, x_k$ and any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|\frac{\#\{m \le n : X_m = x_i\}}{n} - f(x_i)\right| < \varepsilon \ \text{ for all } i \in \{1, \dots, k\}\right) \xrightarrow{n\to\infty} 1.$$

Hint. Use part (a).

7. (10 points) An exam is applied to two classes of 100 students each. The scores of students of class 1 are identically distributed with expectation 7.5 and variance 6. The scores of students of class 2 are identically distributed with expectation 8 and variance 3. Assume that students' scores are all independent. Estimate the probability that the average exam score of class 1 is larger than the average exam score of class 2.